



Functionals of exponential Brownian motion and divided differences

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ABSTRACT

We provide a surprising new application of classical approximation theory to a fundamental asset-pricing model of mathematical finance. Specifically, we calculate an analytic value for the correlation coefficient between the exponential Brownian motion and its time average, and we find that the use of divided differences greatly elucidates formulae, providing a path to several new results. As applications, we find that this correlation coefficient is always at least $1/\sqrt{2}$ and, via the Hermite–Genocchi integral relation, demonstrate that all moments of the time average are certain divided differences of the exponential function. We also prove that these moments agree with the somewhat more complex formulae obtained by Oshanin and Yor.

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1. Introduction

We begin with exponential, or geometric, Brownian motion, defined by

$$S(t) = e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma B(t)}, \quad t \geq 0, \quad (1.1)$$

where r and σ are non-negative constants, and $B : [0, \infty) \rightarrow \mathbb{R}$ is Brownian motion. In other words, B is a stochastic process, or *random function*, for which $B(0) = 0$, its increments are independent, and for $0 \leq s < t$, the increment $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$. The basic properties of Brownian motion are explained in Section 37 of Billingsley [1], while Karatzas and Shreve [2] is a comprehensive treatise.

We shall study the *time average*:

$$A(T) := \frac{1}{T} \int_0^T S(t) dt \quad (1.2)$$

using the calculus of divided differences, a fundamental tool in approximation theory. We will, in particular, show that the correlation coefficient between $A(T)$ and $S(T)$, the moments of $A(T)$ and, more generally, joint moments of $S(T)$ and $A(T)$ can be elegantly, and usefully, expressed in terms of divided differences of the exponential function. Now the time average $A(T)$ has been extensively studied in the literature of Asian options; see, for instance, [3,4]. However, we find that our use of divided differences both simplify and elucidate formulae. In Section 2, we derive the correlation coefficient for $S(T)$ and $A(T)$, finding that it is always at least $1/\sqrt{2}$, thus explaining the relative high correlation that is observational folklore in the financial community. In Section 3, we demonstrate that the divided differences occurring in the lower moments of $S(T)$ and $A(T)$ generalise to all moments, using the fact that the integral of an exponential function over a simplex can be expressed, via the Hermite–Genocchi formula, as a certain divided difference of the exponential function. In Section 4, we provide the

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divided difference theory required by the paper. Finally, in Section 5, we use our divided difference approach to derive a recurrence relation for the moments of $A(T)$.

We first observe the familiar result

$$\mathbb{E}S(T) = e^{(r-\sigma^2/2)T} \mathbb{E}e^{\sigma T^{1/2}Z} = e^{(r-\sigma^2/2)T} e^{\sigma^2 T/2} = e^{rT}. \quad (1.3)$$

Here Z denotes a generic $N(0, 1)$ Gaussian random variable and we have used the standard fact that

$$\mathbb{E}e^{\lambda Z} = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{\lambda \tau} e^{-\tau^2/2} d\tau = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-\frac{1}{2}\{(\tau-\lambda)^2 - \lambda^2\}} d\tau = e^{\lambda^2/2}. \quad (1.4)$$

Similarly,

$$\begin{aligned} \mathbb{E}A(T) &= T^{-1} \int_0^T \mathbb{E}S(t) dt \\ &= \frac{e^{rT} - 1}{rT}. \end{aligned} \quad (1.5)$$

The approximation theorist will immediately recognise the divided difference

$$\mathbb{E}A(T) = \exp[0, rT], \quad (1.6)$$

but a sceptical reader might view this as mere coincidence; in fact, it is but the tip of an iceberg. We remind the reader that $f[a_0, a_1, \dots, a_n]$ is the highest coefficient of the unique polynomial of degree n interpolating f at distinct points $a_0, \dots, a_n \in \mathbb{R}$, which implies $f[a_0] = f(a_0)$ and

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0}.$$

Further, it is evident that a divided difference does not depend on the order in which the points a_0, a_1, \dots, a_n are chosen. As mentioned above, Section 4 collects further divided difference theory required by this paper.

2. The correlation coefficient between the time average and the asset

We shall compute the correlation coefficient between $S(T)$ and $A(T)$. Specifically, we calculate

$$R := \frac{\mathbb{E}(S(T)A(T)) - \mathbb{E}(S(T))\mathbb{E}(A(T))}{\sqrt{\text{var } S(T) \text{ var } A(T)}}. \quad (2.1)$$

We find an elegant divided difference expression for R .

Theorem 2.1. *The correlation coefficient (2.1) is given by*

$$R \equiv R(rT, \sigma^2 T) = \frac{\exp[rT, 2rT, (2r + \sigma^2)T]}{\sqrt{2 \exp[2rT, (2r + \sigma^2)T] \exp[0, rT, 2rT, (2r + \sigma^2)T]}}. \quad (2.2)$$

Let us begin our derivation.

Lemma 2.2. *If $0 \leq a \leq b$, then*

$$\mathbb{E}S(a)S(b) = \exp\left(a(r + \sigma^2) + br\right). \quad (2.3)$$

Proof. We have

$$\begin{aligned} \mathbb{E}S(a)S(b) &= \mathbb{E}S(a)^2 e^{(b-a)(r-\sigma^2/2)+\sigma(B(b)-B(a))} \\ &= \mathbb{E}S(a)^2 \mathbb{E}e^{(b-a)(r-\sigma^2/2)+\sigma\sqrt{b-a}Z} \\ &= e^{(2r+\sigma^2)a} e^{(b-a)r} \\ &= e^{a(r+\sigma^2)} e^{br}, \end{aligned} \quad (2.4)$$

where $Z \sim N(0, 1)$ and we have used (1.3). \square

Proposition 2.3. *We have*

$$\mathbb{E}S(T)A(T) = \exp[rT, (2r + \sigma^2)T]. \quad (2.5)$$

Proof. Applying Lemma 2.2, we obtain

$$\begin{aligned}\mathbb{E}S(T)A(T) &= T^{-1} \int_0^T \mathbb{E}S(t)S(T) dt \\ &= T^{-1} \int_0^T e^{(r+\sigma^2)t} e^{rT} dt \\ &= \exp[rT, (2r + \sigma^2)T]. \quad \square\end{aligned}$$

Proposition 2.4.

$$\mathbb{E}(A(T)^2) = 2 \exp[0, rT, (2r + \sigma^2)T]. \quad (2.6)$$

Proof. We find

$$\begin{aligned}\mathbb{E}(A(T)^2) &= T^{-2} \int_0^T \left(\int_0^T \mathbb{E}S(t_1)S(t_2) dt_2 \right) dt_1 \\ &= 2T^{-2} \int_0^T \left(\int_0^{t_1} \mathbb{E}S(t_1)S(t_2) dt_2 \right) dt_1.\end{aligned} \quad (2.7)$$

Thus

$$\begin{aligned}\mathbb{E}(A(T)^2) &= 2T^{-2} \int_0^T \left(\int_0^{t_1} e^{r(t_1+t_2)} e^{\sigma^2 t_2} dt_2 \right) dt_1 \\ &= 2T^{-2} \int_0^T e^{rt_1} \left(\frac{e^{(r+\sigma^2)t_1} - 1}{r + \sigma^2} \right) dt_1 \\ &= \frac{2}{(r + \sigma^2)T} [\exp[0, (2r + \sigma^2)T] - \exp[0, rT]] \\ &= 2 \exp[0, rT, (2r + \sigma^2)T],\end{aligned} \quad (2.8)$$

using the divided difference recurrence relation (4.1) to obtain the final line. \square

Any reader still doubtful of the simplification provided by divided difference notation might consider the alternative expression provided in [5]:

$$\mathbb{E}(A(T)^2) = \frac{2e^{(2r+\sigma^2)T}}{(r + \sigma^2)(2r + \sigma^2)T^2} + \frac{2}{rT^2} \left(\frac{1}{2r + \sigma^2} - \frac{e^{rT}}{r + \sigma^2} \right).$$

There is a similar divided difference relation for $\mathbb{E}(A(T)^m)$, described in the next section, but we now complete our derivation of Theorem 2.1.

Proof of Theorem 2.1. Applying (1.4), (1.5), (2.5) and (4.1), we obtain

$$\begin{aligned}\mathbb{E}S(T)A(T) - \mathbb{E}S(T)\mathbb{E}A(T) &= \exp[rT, (2r + \sigma^2)T] - e^{rT}(e^{rT} - 1)/(rT) \\ &= \exp[rT, (2r + \sigma^2)T] - \exp[rT, 2rT] \\ &= \sigma^2 T \exp[rT, 2rT, (2r + \sigma^2)T].\end{aligned} \quad (2.9)$$

Further,

$$\text{var } S(T) = \mathbb{E}(S(T)^2) - (\mathbb{E}S(T))^2 = e^{(2r+\sigma^2)T} - e^{2rT} = \sigma^2 T \exp[2rT, (2r + \sigma^2)T], \quad (2.10)$$

and, by (1.5) and (2.6),

$$\begin{aligned}\text{var } A(T) &= 2 \exp[0, rT, (2r + \sigma^2)T] - \left(\frac{e^{rT} - 1}{rT} \right)^2 \\ &= 2 \exp[0, rT, (2r + \sigma^2)T] - 2 \exp[0, rT, 2rT] \\ &= 2\sigma^2 T \exp[0, rT, 2rT, (2r + \sigma^2)T],\end{aligned} \quad (2.11)$$

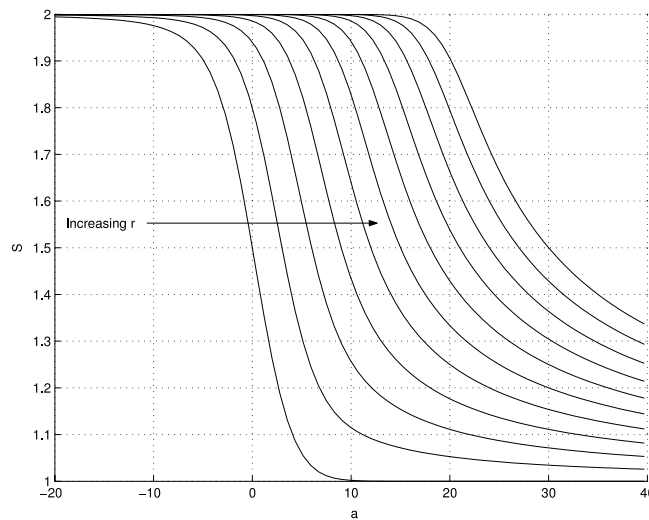


Fig. 1. $S(r, a)$ for $-20 \leq a \leq 40$ and $0.1 \leq r \leq 10$.

using the divided difference recurrence (4.1) once more. Hence

$$R = \frac{\exp[rT, 2rT, (2r + \sigma^2)T]}{\sqrt{2 \exp[2rT, (2r + \sigma^2)T] \exp[0, rT, 2rT, (2r + \sigma^2)T]}}. \quad \square \quad (2.12)$$

It is remarkable that the divided differences appearing in (2.12) are coefficients of the cubic polynomial interpolating the exponential function at $0, rT, 2rT, (2r + \sigma^2)T$. We make three further observations:

- (1) Armed with an analytic expression for the correlation coefficient, we can apply the exchange option valuation formula of Margrabe [6] to derive the values of certain Asian options, if we are willing to accept that the time average is suitably approximated by exponential Brownian motion. We are investigating the numerics of this rather simple approximation at present and preliminary results are surprisingly promising.
- (2) The correlation coefficient $R(rT, \sigma^2 T)$ is typically close to unity: typical values of r, σ and T produce values of R in the 0.8–0.9 range. In fact, we are able to prove that the correlation coefficient satisfies $R(rT, \sigma^2 T) \geq 1/\sqrt{2}$, for all $r \geq 0, \sigma \geq 0$ and $T > 0$, a surprisingly high lower bound for the correlation coefficient. The details of this derivation are too complicated to include here, and we refer the reader to Baxter and Fretwell [7] for further details. However, the numerical findings are summarised in Fig. 1, which displays values of the closely related quantity

$$S \equiv S(r, a) = \frac{(\exp[a, 2r, r])^2}{\exp[a, 2r] \exp[a, 2r, r, 0]},$$

for $-20 \leq a \leq 40$ and $0.1 \leq r \leq 10$. It is easily checked that

$$R(rT, \sigma^2 T) = \sqrt{S(rT, (2r + \sigma^2)T)/2},$$

so that the lower bound $R \geq 1/\sqrt{2}$ becomes $S \geq 1$. It is plausible that $S(r, a)$ should be a decreasing function of a , for fixed r , because correlation should be a decreasing function of volatility. Further, it is not difficult to establish the limiting values $\lim_{a \rightarrow -\infty} S(a, r) = 2$ and $\lim_{r \rightarrow \infty} S(a, r) = 1$. However, further analysis is not straightforward, and the analysis of Baxter and Fretwell [7] makes great use of properties of divided differences.

- (3) It is natural to ask whether these divided difference expressions are particular to exponential Brownian motion. In fact, similar expressions occur when exponential Brownian motion is replaced by certain Lévy-stable variants: see [7] for further details.

3. Computing higher moments of $A(T)$

We now demonstrate that the neat divided difference formulae obtained for the first and second moments of $A(T)$ are *not* coincidences, but part of a greater pattern from which arise new formulae generalising the moment calculations of Oshanin et al. [4] and Yor [3].

We begin with the iterated integral

$$\mathbb{E}A(T)^m = T^{-m} \int_0^T d\tau_m \int_0^T d\tau_{m-1} \dots \int_0^T d\tau_1 \mathbb{E}S(\tau_1) \dots S(\tau_m). \quad (3.1)$$

Now, given any point $(\tau_1, \dots, \tau_m) \in [0, T]^m$, let us sort its components into increasing order, obtaining (t_1, \dots, t_m) (say). Then

$$\mathbb{E}S(\tau_1) \cdots S(\tau_m) = \mathbb{E}S(t_1) \cdots S(t_m)$$

and

$$\mathbb{E}A(T)^m = m!T^{-m} \int_0^T dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \mathbb{E}S(t_1) \cdots S(t_m). \quad (3.2)$$

Our first task is to calculate the integrand, which we complete after a simple lemma.

Lemma 3.1. For any positive integer k , we have

$$\mathbb{E}[S(t)^k] = \exp\left(krt + \frac{\sigma^2 t}{2}k(k-1)\right). \quad (3.3)$$

Proof. This is almost immediate from (1.4):

$$\mathbb{E}S(t)^k = \mathbb{E}e^{k(r-\sigma^2/2)t + \sigma k\sqrt{t}Z} = e^{k(r-\sigma^2/2)t + \sigma^2 k^2 t/2} = e^{krt + \sigma^2 tk(k-1)/2},$$

where $Z \sim N(0, 1)$. \square

Proposition 3.2. If $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m$, then

$$\mathbb{E}S(t_1)S(t_2) \cdots S(t_m) = \exp\left(\sum_{k=1}^m (r + (m-k)\sigma^2) t_k\right). \quad (3.4)$$

Proof. Lemma 2.2 comprises the case $m = 2$. We complete the proof by induction on the number of terms m , first observing that, by a standard property of geometric Brownian motion,

$$\mathbb{E}S(t_1)S(t_2) \cdots S(t_m) = \mathbb{E}S(t_1)^m \mathbb{E}S(t_2 - t_1) \cdots S(t_m - t_1). \quad (3.5)$$

Applying Lemma 3.1 and our induction hypothesis, we obtain

$$\mathbb{E}S(t_1)S(t_2) \cdots S(t_m) = \exp\left(mrt_1 + \sigma^2 t_1 m(m-1)/2 + \sum_{\ell=2}^m (r + (m-\ell)\sigma^2) (t_\ell - t_1)\right). \quad (3.6)$$

The t_1 coefficient in the exponent is given by

$$mr - (m-1)r + \sigma^2 t_1 \left(\frac{1}{2}m(m-1) - \sum_{\ell=1}^{m-2} \ell\right) = r + \sigma^2 t_1(m-1),$$

using the elementary fact that $m(m-1)/2 = 1 + 2 + \cdots + m-1$. The coefficients of t_2, \dots, t_m are as already stated in (3.4). \square

Thus the desired integral (3.2) becomes

$$\begin{aligned} \mathbb{E}A(T)^m &= m!T^{-m} \int_0^T dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \mathbb{E}S(t_1) \cdots S(t_m) \\ &= m! \int_0^1 dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \exp(\alpha_1 t_1 + \cdots + \alpha_m t_m), \end{aligned} \quad (3.7)$$

where

$$\alpha_k = (r + (m-k)\sigma^2)T, \quad k = 1, \dots, m. \quad (3.8)$$

The integral displayed in (3.7) can now be identified as a divided difference using a variant form of the Hermite–Genocchi integral relation.

Theorem 3.3. Let

$$b_k := kr + \sigma^2 k(k-1)/2, \quad k = 0, 1, \dots \quad (3.9)$$

Then

$$\mathbb{E}(A(T))^m = m! \exp[b_0 T, b_1 T, \dots, b_m T], \quad m \geq 0. \quad (3.10)$$

Proof. Apply Corollary 4.5 to (3.7) and (3.8), using $\sum_{k=1}^j k = j(j+1)/2$. \square

The statement of [Theorem 3.3](#) simplifies when $r = \sigma^2$, for then the drift term in (1.1) vanishes, that is, we consider $S(t) = \exp(\sigma \sqrt{t}B(t))$ alone; this is the special case studied by Oshanin et al. [4] and Yor [3], for the formulae grow *much* more complicated without the use of divided differences. Therefore we now demonstrate that our expression agrees with theirs.

Theorem 3.4. *If we set $r = \sigma^2/2$ in [Theorem 3.3](#), then we obtain*

$$\begin{aligned}\mathbb{E}[A(T)^m] &= m! \exp[0, rT, 2^2 rT, 3^2 rT, \dots, m^2 rT] \\ &= m! H_{\sqrt{rT}}[-m, \dots, -1, 0, 1, \dots, m],\end{aligned}\quad (3.11)$$

where $H_c(x) := \exp(c^2 x^2)$, $x \in \mathbb{R}$, for any positive c .

Proof. We simply set $r = \sigma^2/2$ in [Theorem 3.3](#) and apply (4.12). \square

We can now apply [Corollary 4.10](#) to derive the formula given in Eq. (14) of Oshanin et al. [4].

Theorem 3.5. *If we set $r = \sigma^2/2$, then*

$$\mathbb{E}[A(T)^m] = \left(\frac{\Gamma(m)}{\Gamma(2m)} \right) r^{-m} \left(-\frac{1}{2} (-1)^m \binom{2m}{m} + \sum_{\ell=0}^m \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2} \right). \quad (3.12)$$

Proof. Applying [Corollary 4.10](#) to [Theorem 3.4](#), we obtain

$$\begin{aligned}\mathbb{E}[A(T)^m] &= \left(\frac{m!}{(2m)!} (rT)^{-m} \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k e^{rT(k-m)^2} \right) \\ &= \left(\frac{\Gamma(m)}{\Gamma(2m)} \right) (rT)^{-m} \left(-\frac{1}{2} (-1)^m \binom{2m}{m} + \sum_{\ell=0}^m \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2} \right),\end{aligned}\quad (3.13)$$

after some straightforward algebraic manipulation. \square

If we now replace rT by α and m by j in (3.12), then we obtain Eq. (14) of Oshanin et al. [4].

4. Divided difference theory

Most of the properties of divided differences required here can be found in Chapter 5 of Powell [8] or Chapter 1 of de Boor [9]. There is also much useful material in Chapter 7 of the treatise of DeVore and Lorentz [10], or indeed the original Hermite [11].

We recall the *divided difference recurrence relation*.

Theorem 4.1.

$$f[a_0, a_1, \dots, a_n] = \frac{f[a_1, \dots, a_n] - f[a_0, \dots, a_{n-1}]}{a_n - a_0}, \quad (4.1)$$

for any distinct complex numbers a_0, \dots, a_n .

Proof. See, for instance, [8, Theorem 5.3]. \square

If f is sufficiently differentiable, then we can, of course, define divided differences for coincident points. Further, the elementary relation

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = \int_0^1 f'((1-t)a_0 + ta_1) dt, \quad \text{when } a_0, a_1 \in \mathbb{R}, \quad (4.2)$$

can be generalised to obtain the *Hermite–Genocchi formula*.

Theorem 4.2 (Hermite–Genocchi). *Let $f \in C^{(n)}(\mathbb{R})$ and let a_0, a_1, \dots, a_n be (not necessarily distinct) real numbers. Then, for $n \geq 1$,*

$$\begin{aligned}f[a_0, a_1, \dots, a_n] &= \int_{S_n} f^{(n)}(t_0 a_0 + t_1 a_1 + \dots + t_n a_n) dt_1 \dots dt_n, \\ &= \int_0^1 dt_1 \int_0^{1-t_1} dt_2 \dots \int_0^{1-\sum_{k=1}^{n-1} t_k} dt_n f^{(n)}(t_0 a_0 + t_1 a_1 + \dots + t_n a_n)\end{aligned}\quad (4.3)$$

where the domain of integration is the simplex

$$S_n = \left\{ t = (t_1, t_2, \dots, t_n) \in \mathbb{R}_+^n : \sum_{k=1}^n t_k \leq 1 \right\} \quad (4.4)$$

and

$$t_0 = 1 - \sum_{k=1}^n t_k.$$

Proof. A laconic derivation is given in Chapter 4, Section 7(k) of DeVore and Lorentz [10]. \square

We shall need a variant form of the Hermite–Genocchi integral relation for which the following notation is useful. Given any real $n \times n$ nonsingular matrix V , with columns v_1, \dots, v_n , we let $K(V)$ denote the closed convex hull of $0, v_1, \dots, v_n$, i.e.

$$K(V) := \text{conv}\{0, v_1, \dots, v_n\}.$$

In this notation, the Hermite–Genocchi integral relation states that

$$f[a_0, a_1, \dots, a_n] = \int_{K(I_n)} f^{(n)}(a_0 + (a - a_0 e)^T y) \, dy, \quad (4.5)$$

where

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and I_n denotes the $n \times n$ identity matrix. Integrating the n th derivative over the simplex $K(V)$ yields a useful variant form of Hermite–Genocchi.

Theorem 4.3. Let $V \in \mathbb{R}^{n \times n}$ be any nonsingular matrix. Then

$$\frac{1}{|\det V|} \int_{K(V)} f^{(n)}(a^T y) \, dy = f[0, (V^T a)_1, \dots, (V^T a)_n], \quad (4.6)$$

where $(V^T a)_k$ denotes the k th component of the vector $V^T a$.

Proof. Substituting $y = Vz$, Hermite–Genocchi implies the relation

$$\int_{K(V)} f^{(n)}((V^T a)^T z) \, dz = f[0, (V^T a)_1, \dots, (V^T a)_n]. \quad \square$$

Corollary 4.4. For any function $f \in C^{(n)}(\mathbb{R})$, we have

$$\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \dots \int_0^{x_2} dx_1 f^{(n)}\left(\sum_{k=1}^n a_k x_k\right) = f[0, a_n, a_n + a_{n-1}, \dots, a_n + a_{n-1} + \dots + a_1]. \quad (4.7)$$

Proof. Set

$$V = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

in Theorem 4.3. \square

The exponential function is a particularly important case for us, in which case the Hermite–Genocchi formula becomes

$$\exp[a_0, a_1, \dots, a_n] = \int_{S_n} e^{t_0 a_0 + t_1 a_1 + \dots + t_n a_n} dt_1 \dots dt_n \quad (4.8)$$

and Corollary 4.4 takes the following form.

Corollary 4.5. We have

$$\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \dots \int_0^{x_2} dx_1 \exp\left(\sum_{k=1}^n a_k x_k\right) = \exp[0, a_n, a_n + a_{n-1}, \dots, a_n + a_{n-1} + \dots + a_1]. \quad (4.9)$$

Proof. Let f be the exponential function in Corollary 4.4. \square

Further, we note that, for the exponential function, Theorem 4.3 becomes the interesting formula

$$\frac{1}{|\det V|} \int_{K(V)} e^{a^T y} dy = \exp[0, (V^T a)_1, \dots, (V^T a)_n]. \quad (4.10)$$

Thus, integrating exponentials over simplexes or, more generally, a polyhedron formed by the disjoint union of simplexes, will generate divided differences of the exponential.

We shall also need two simple preliminary results. Let us use \mathbb{P}_n to denote the vector space of polynomials of degree n .

Lemma 4.6. We have

$$\exp(\mu) \exp[\lambda_0, \dots, \lambda_m] = \exp[\lambda_0 + \mu, \dots, \lambda_m + \mu], \quad (4.11)$$

where $\lambda_0, \dots, \lambda_m$ and μ can be any complex numbers.

Proof. Immediate. \square

Lemma 4.7. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ and let a_1, \dots, a_n be distinct nonzero complex numbers. Then

$$f[0, a_1^2, \dots, a_n^2] = g[-a_n, \dots, -a_1, 0, a_1, \dots, a_n], \quad (4.12)$$

where $g(z) = f(z^2)$, for $z \in \mathbb{C}$.

Proof. Let $p \in \mathbb{P}_n$ interpolate f at $0, a_1^2, \dots, a_n^2$. Then $q(z) := p(z^2)$ is a polynomial of degree $2n$ satisfying $q(\pm a_j) = p(a_j^2) = f(a_j^2) = g(\pm a_j)$, for $j = 0, \dots, n$, setting $a_0 = 0$, for convenience. The result then follows from uniqueness of the interpolating polynomial. \square

It is well known that a divided difference at equally spaced points can be expressed in a particularly simple form using the forward difference operator

$$\Delta_h f(x) := f(x+h) - f(x),$$

which we shall need when demonstrating the equivalence between our moment calculations and those of Oshanin et al. [4] and Yor [3]. The next proposition should be well-known, but we include its short proof for the reader's convenience.

Proposition 4.8. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, let h be any positive constant and let n be a non-negative integer. Then

$$f[x, x+h, x+2h, \dots, x+nh] = \frac{\Delta_h^n f(x)}{n!h^n}. \quad (4.13)$$

Proof. It is easily checked that $f[x, x+h] = \Delta_h f(x)/h$. Further, if we assume (4.13) for $n-1$, then the divided difference recurrence relation implies that

$$\begin{aligned} f[x, x+h, \dots, x+nh] &= \frac{f[x+h, \dots, x+nh] - f[x, x+h, \dots, x+(n-1)h]}{nh} \\ &= \frac{\Delta_h f[x, \dots, x+(n-1)h]}{nh} \\ &= \frac{1}{nh} \Delta_h \left(\frac{\Delta_h^{n-1} f(x)}{(n-1)!h^{n-1}} \right) \\ &= \frac{\Delta_h^n f(x)}{n!h^n}. \end{aligned}$$

Thus the result follows by induction. \square

Corollary 4.9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and let h be any positive constant. Then

$$f[x, x+h, x+2h, \dots, x+nh] = \frac{1}{n!h^n} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(x+kh). \quad (4.14)$$

Proof. We define the forward shift operator

$$E_h f(x) := f(x+h), \quad x \in \mathbb{R},$$

and observe that, by the binomial theorem,

$$\Delta_h^n f(x) = (E_h - 1)^n f(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} E_h^k f(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(x+kh). \quad \square$$

Corollary 4.10. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and let h be any positive number. Then

$$f[-nh, -(n-1)h, \dots, -h, 0, h, \dots, nh] = \frac{1}{(2n)!h^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k f((k-n)h). \quad (4.15)$$

Proof. This is an immediate consequence of Corollary 4.9. \square

We shall also need the *Leibniz* relation for divided differences of a product when deriving the recurrence differential equation for moments.

Theorem 4.11 (Leibniz). Let D be any subset of \mathbb{C} containing the distinct points z_0, z_1, \dots, z_n and let v and w be complex-valued functions on D . If $u = v \cdot w$, then

$$u[z_0, \dots, z_n] = \sum_{k=0}^n v[z_0, \dots, z_k] w[z_k, \dots, z_n]. \quad (4.16)$$

Proof. See, for instance, Chapter 1, art. (iv) of de Boor [9]. \square

5. A recurrence relation

The Feynman–Kac formula (see, for example, [2]) suggests that the moments $E_n(t) := \mathbb{E}(A(t)^n)$ of the time average should satisfy a certain differential equation, which we shall also obtain as an illustration of the divided difference approach.

Theorem 5.1. Let $\{c_n\}_{n=1}^\infty$ be any strictly increasing sequence of positive numbers and define $e_n: (0, \infty) \rightarrow \mathbb{R}$ by the divided difference

$$e_n(t) = \exp[0, c_1 t, \dots, c_n t], \quad t > 0, \quad n \geq 0. \quad (5.1)$$

Then

$$te'_n(t) = e_n(t)(c_n t - n) + e_{n-1}(t), \quad \text{for } n \geq 1. \quad (5.2)$$

Proof. Applying the Hermite–Genocchi formula, we obtain

$$e_n(t) = \int_{K(I_n)} \exp(tc^T y) \, dy, \quad (5.3)$$

where $b = (c_1, \dots, c_n)^T$, and differentiating (5.3) yields

$$e'_n(t) = \int_{K(I_n)} \exp(tc^T y)(c^T y) \, dy. \quad (5.4)$$

Now writing $g(s) = s$ and applying Leibniz's formula for divided differences, we find

$$\begin{aligned} (g \cdot \exp)[0, c_1 t, \dots, c_n t] &= g[0] \exp[0, c_1 t, \dots, c_n t] + g[0, c_1 t] \exp[c_1 t, \dots, c_n t] \\ &= \exp[c_1 t, \dots, c_n t]. \end{aligned} \quad (5.5)$$

Further, the relation $(g \cdot \exp)^{(n)} = g \cdot \exp + n \exp$ and (5.5) imply

$$\begin{aligned} te'_n(t) &= \int_{K(I_n)} (g \cdot \exp)^{(n)}(tc^T y) \, dy - n \int_{K(I_n)} \exp(tc^T y) \, dy \\ &= (g \cdot \exp)[0, c_1 t, \dots, c_n t] - n \exp[0, c_1 t, \dots, c_n t] \\ &= \exp[c_1 t, \dots, c_n t] - ne_n(t). \end{aligned} \quad (5.6)$$

However,

$$e_n(t) = \frac{\exp[c_1 t, \dots, c_n t] - e_{n-1}(t)}{c_n t}, \quad (5.7)$$

by the divided difference recurrence relation, so that

$$\exp[c_1 t, \dots, c_n t] = c_n te_n(t) + e_{n-1}(t). \quad (5.8)$$

Substituting (5.8) in (5.6) provides (5.2). \square

The corresponding differential equation for E_n is now immediate.

Corollary 5.2. *The moments satisfy*

$$tE'_n(t) = E_n(t) (b_n t - n) + E_{n-1}(t), \quad \text{for } n \geq 1, \quad (5.9)$$

where b_n is given by (3.9).

Proof. We apply Theorems 5.1 and 3.3. \square

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